

# On Braid Excitations in Quantum Gravity

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## Abstract

We propose a new notation for the states in some models of quantum gravity, namely 4-valent spin networks embedded in a topological three manifold. With the help of this notation, equivalence moves, namely translations and rotations, can be defined, which relate the projections of diffeomorphic embeddings of a spin network. Certain types of topological structures, viz 3-strand braids as local excitations of embedded spin networks, are defined and classified by means of the equivalence moves. This paper formulates a mathematical approach to the further research of particle-like excitations in quantum gravity.

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# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>Notation</b>	<b>3</b>
2.1	Framed and unframed spin networks . . . . .	5
<b>3</b>	<b>Braids</b>	<b>5</b>
<b>4</b>	<b>Equivalence Moves</b>	<b>7</b>
4.1	Translation Moves . . . . .	9
4.2	Rotations . . . . .	9
4.2.1	$\pi/3$ -Rotations . . . . .	9
4.2.2	$2\pi/3$ -Rotations . . . . .	10
4.2.3	$\pi$ -Rotations . . . . .	10
4.3	Conserved quantity: effective twist number . . . . .	14
<b>5</b>	<b>Classification of Braids</b>	<b>16</b>
<b>6</b>	<b>Conclusions &amp; Perspectives</b>	<b>24</b>

# 1 Introduction

Loop quantum gravity had never been considered a candidate of the unification of matter and gravity until a remarkable series of discoveries emerged recently. First, Markopoulou and Kribs[1] discovered that loop quantum gravity and many related theories of dynamical quantum geometry have emergent excitations which carry conserved quantum numbers not associated with geometry or the gravitational field.

Around the same time, Bilsen-Thompson[2] found that a composite or “preon” model of the quarks, leptons and vector bosons could be coded in the possible ways that three ribbons can be braided and twisted. This suggested that the particles of the standard model could be discovered amidst the emergent braid states and their conserved quantum numbers associated with those of the standard model. One realization of this was then given in [3], for a particular class of dynamic quantum geometry models based on 3-valent quantum spin-networks obtained by gluing trinions together. These are coded in the knotting and braiding of the edges of the spin network; they are degrees of freedom because of the basic result that quantum gravity or the quantization of any diffeomorphism invariant gauge theory has a basis of states given by embeddings up to diffeomorphisms of a set of labeled graphs in a spatial manifold. Indeed, the role of the braiding of the edges of the graphs had been a mystery for many years.

However, spin foam models in  $3 + 1$  dimensions involve embedded 4-valent spin networks[4]. It is then natural to ask if there are conservation laws associated with braids in 4-valent spin-networks. Besides, quantum gravity with a positive cosmological constant[6] and quantum deformation of quantum gravity[5] suggest the framing of embedded spin-networks. In this paper we extend the investigation of the braid excitations from the 3-valent case to the 4-valent case. We study (framed) 4-valent spin-networks embedded in 3D.

Due to the complexity of embedded 4-valent spin-networks, to deal with the braid excitations of them we need a consistent and convenient mathematical formalism. In this paper, which is the first of a series of papers on the subject, we first propose a new notation of the embedded (framed) 4-valent spin-networks and define what we mean by braids, then discuss equivalence moves with the help of our notation, which relate all diffeomorphic embedded 4-valent graphs and form the graphical calculus of the kinematics of these graphs, and at the end present a classification of the braids. These results are key to our subsequent papers. We focus on 3-strand braids, which are the simplest non-trivial and interesting braid excitations living on embedded 4-valent spin-networks.

## 2 Notation

Firstly, we fix the notation, namely a tube-sphere notation. We work in the category of framed graphs, in particular the two dimensional projections representing embedded

framed 4-valent spin networks up to diffeomorphisms. There is a single diffeomorphism class of nodes. We therefore represent nodes by rigid 2-spheres and edges by tubes. Such a node can be considered locally dual to a tetrahedron, as shown in Fig. 1(a). If the spin-nets are not framed, we simply reduce tubes to lines but still keep spheres as nodes. To fully characterize the embedding of a spin-net in a 3-manifold, we assume that not only the nodes are rigid, i.e. they can only be rotated or translated, but also the positions on the node where the edges are attached are fixed. This requirement and the local duality ensures the non-degeneracy of the nodes, i.e. no more than two edges of a node are coplanar. For the convenience of calculation, we simplify the tube-sphere notation in Fig. 1(a) to Fig. 1(b), in which 1) the sphere is replaced by a solid circle; 2) the two tubes in the front,  $A$  and  $C$  in (a), are replaced by a solid line piercing through the circle in (b); and 3) the two tubes in the back,  $B$  and  $D$  in (a) are substituted by  $B$  and  $D$  in (b) with a dashed line connecting them through the circle. There is no loss of generality in taking this simplified notation, because one can always arrange a node in the two states like Fig. 1(b) & (c) by diffeomorphism before taking a projection. Due to the local duality between a node and a tetrahedron and the fact that all the four edges of a node are on an equal footing, if we choose one of the four edges of a node at a time, the other three edges are still on an equal footing, in respect to a rotation symmetry with the specially chosen edge as the rotation axis, e.g. the edge  $B$  in Fig. 1(b) & (c). This rotation symmetry will be discussed in detail in the next section.

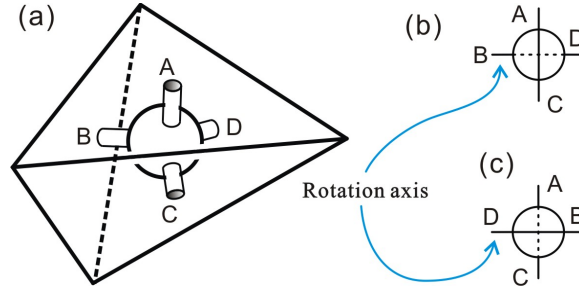


Figure 1: (a) is a tetrahedron and its dual node. There are two orientations of a node that can appear in a diagram, (b) denotes the  $\oplus$  state, while (c) denotes the  $\ominus$  state.

There could exist twists on embedded tubes, e.g. the  $\pi/3$ -twist on the edge  $B$  with respect to the solid red dot, shown in Fig. 2(a). Note that we put twists in the unit of  $\pi/3$  for two reasons. The first reason is that the possible states by which a node may be represented in a projection can be taken into each other by  $\pi/3$  rotations around one of the edges of the node (this will become clear in Section 4.2). By the local duality of a node to a tetrahedron, these correspond to the  $\pi/3$  rotations that relate the different ways that two tetrahedra may be glued together on a triangular face. These rotations create twists in the edges and, as a result of the restriction on projections of nodes we impose, set the twists in a projection of an edge of a spin network in units of  $\pi/3$ . The other reason is that

the least twist distinguishable from zero of a piece of tube in a projection is  $\pi/3$  and all higher twists distinguishable from each other in the projection must then be multiples of  $\pi/3$ .

Because an edge is always between two nodes and a rotation of a node creates/annihilates twists on its edges, one usually needs to specify the fixed point on an edge with respect to which a twist is counted, as shown in Fig. 2(a). In this manner, the 1 unit of twist in Fig. 2(a) is obviously equivalent to that in 2(b), which is the same amount of twist in the opposite direction on the other side of the fixed point. Interestingly, both twists in Fig. 2(a) and (b) are right-handed twists if one points his/her right thumb to the node on the same sides of the fixed point as that of the twists; therefore, we can unambiguously assign the same value to them, namely  $+1$  (unit of  $\pi/3$ ). This provides a way of simplifying the notation of twist, i.e. we can simply label an edge with a (left-) right-handed twist a (negative) positive integer. For example, Fig. 2(a) and (b) can be replaced by 2(c) without ambiguity.

Recalling the rotation axis mentioned before, one can assign states to a node with respect to its rotation axis. If the rotation axis is an edge in the back, we say the node is in state  $\oplus$ , or is simply called a  $\oplus$ -node, e.g. Fig. 1(b) with edge  $B$  as the rotation axis. Otherwise, if the rotation axis is an edge in the front, the node is called a  $\ominus$ -node or in the state  $\ominus$ , e.g. Fig. 1(c) with edge  $D$ .

## 2.1 Framed and unframed spin networks

The results of this paper will refer to the case of framed spin networks, defined above. However, unframed graphs are used in loop quantum gravity and it is useful to have results then for that case as well. The particular notation of unframed graphs is obtained from the framed case discussed here by dropping information about twists of the edges (which thus represent curves rather than tubes), but keeping the nodes as rigid spheres, locally dual to tetrahedra. This is necessary so that the evolution moves are well defined for unframed embedded graphs, which will be explained in the second of this series of papers.

In the rest of this paper we refer always to the framed case. Results for the unframed case will be understood from those for the framed case by neglecting the twists of the edges, unless we explicitly describe them.

## 3 Braids

Equipped with the notation defined above, we are interested in a type of topological structures as sub-structures of embedded 4-valent spin networks, namely 3-strand braids, which are defined as follows.

**Definition 1.** *A 3-strand braid (or a braid for simplicity) is a sub-spinnet of an embedded 4-valent spin network, which is a three dimensional object formed by two nodes with three common*

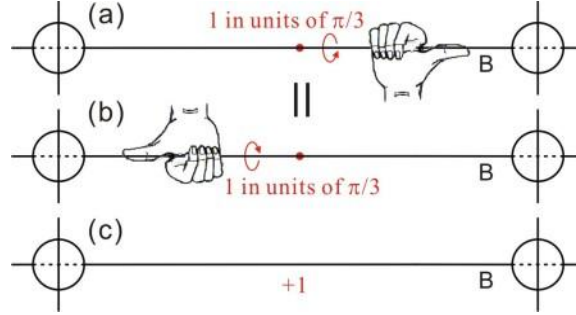


Figure 2: The 1 unit of twist (equivalent to a  $\pi/3$  rotation) in (a) means cut to the right of the red dot and twist as shown. This is equivalent to the opposite twist on the opposite side of the red dot, as shown in (b). Thus, both may be represented as in (c), by a label of  $+1$  of edge  $B$ .

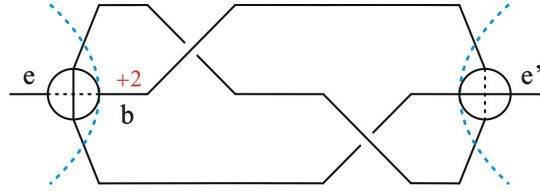


Figure 3: A typical 3-strand braid formed by the three common edges of two end-nodes. The region between the two dashed line satisfies the definition of an ordinary braid. Edges  $e$  and  $e'$  are called external edges. There is also a right handed twist of 2 units on strand  $b$ . In this figure the left handed node is in a  $\oplus$  state while the right handed node is in a  $\ominus$  state.

edges, now named **strands**; the two nodes are called **end-nodes**, each of which has one and only one free edge, called an **external edge**. The two dimensional projections of these braids denoted in our notation are called **braid diagrams**, a typical example of which is shown in Fig. 3. The following conditions should be satisfied:

1. if braids are arranged horizontally, then the (left) right external edge of a braid can always be the (left-) right-most edge of the (left) right end-node, and always stretches to the (left) right, which has no tangles with the strands, for the left part of the braid diagram in Fig. 4(a) as an example;
2. what is captured between the two end-nodes, e.g. the region between the two dashed lines in Fig. 3, should meet the definition of braid in the ordinary braid theory, for the braid diagram in Fig. 4(b) as an example;
3. the three strands of a braid are never tangled with any other edge of the spin-net, as illustrated in right side of the braid diagram in Fig. 4(a), for example.

We would like to emphasize that the braids defined above are 3D structures, each of which has many diffeomorphic embeddings that are represented by their 2D projections in our notation. As a result, the 2D projections of many braids, i.e. their braid diagrams, which appear to be different are actually equivalent to each other in the sense of diffeomorphism. The precise set of equivalence relations will be the topic of the next section. Bearing this in mind, in the rest of the paper we are not going to distinguish braids from their braid diagrams, unless an ambiguity arises.

These kinds of braids are different from the braids in the context of ordinary braid theory, since the two end-nodes of such a braid are topologically significant to the state of the braid. These braid are stable under a certain stability condition regarding the evolution of spin-nets, which will be brought up in the companion paper. However, in this paper, we focus only on the intrinsic properties of these braids, or in other words the pure topological properties of the braids up to diffeomorphism, i.e. without dynamic evolution. To do so, we need to first describe the non-dynamical operations that can be applied to the embedded 4-valent spin networks.

We can assign a number to a crossing according to its chirality, viz +1 for a right-handed crossing, -1 for a left-handed crossing, and 0 otherwise. Fig. 5 shows this assignment.

Such a scheme of assignment will become useful in the subsequent discussions.

## 4 Equivalence Moves

As aforementioned, the tube diagrams of an embedded spin network belong to different equivalence classes. It is therefore obligatory to characterize these equivalence classes by

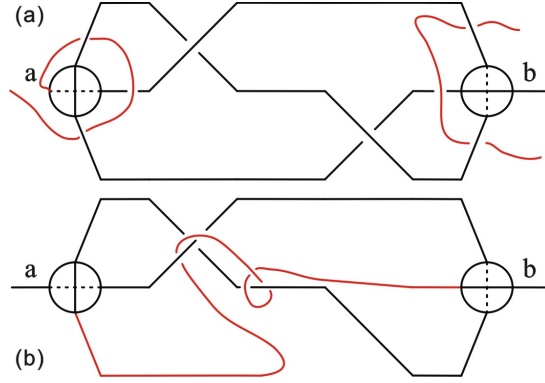


Figure 4: (a) is not a braid due to the tangle between the external edge  $a$  and the common edges of the two nodes, and/or the tangle between the common edges and another edge connecting elsewhere in the whole spinnet. (b) is not a braid either because the region captured between the two nodes does not satisfy the ordinary definition of a braid.

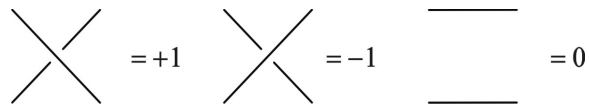


Figure 5: The assignments of right-handed crossing, left-handed crossing, and null crossing respectively from left to right.



equivalence relations. To do so, one needs to find the full set of local moves, operating on the nodes and edges, which don't change the diffeomorphism class of the embedding of a diagram. In the discussion below we work in the framed case. In the unframed case, one just ignores the twists.

An obvious set of equivalence moves consists of the usual three Reidemeister moves, framed or unframed, whose details are not repeated here; these moves will be applied without further notice. More importantly, there are two kinds of equivalence moves that can be peculiarly defined on an embedded 4-valent spin-net, under which two diagrams, in particular two braids, that are related by a sequence of equivalence moves are thought to be equivalent. The first kind composes of translation moves. The second type of equivalence moves are rotations defined on the nodes.

## 4.1 Translation Moves

We discuss translation moves first. Translation moves, which are in fact extended Reidemeister type moves, involve not only the edges but also the nodes of an embedded spin-net; they reflect the translation symmetry of the embedded spin-nets. Let us look at the simplest example first. Fig. 6(a) shows a node  $X$  connected to other places of the network via its four edges; red points represent attached points on other nodes. One can slide the node  $X$  along its edge  $a$  to the left, which leads to Fig. 6(b); this does not change anything of the topology of the embedded spin-net. Fig. 7 illustrates more complicated cases where a crossing is taken into account.

In Fig. 7(a1) there is a node  $X$  and a crossing; however, since the crossing is between the edge  $a$  of node  $X$  and the edge  $e$  of some other node and node  $X$  together with all its edges are above edge  $e$ , one can safely translate node  $X$  along edge  $a$  to the left passing the crossing, which results in Fig. 7(a2), in which the crossing turns out to be between edge  $e$  and edge  $b$ . This, which is obviously a symmetry, may be understood as a Reidemeister move II.

## 4.2 Rotations

Apart from the translation symmetry, there is also a rotation symmetry that gives rise to rotations defined on a node, with respect to one of its four edges, of an embedded spin-net. These rotations are not those with rigid metric but only the ones that change projections of an embedding, without affecting diffeomorphisms.

### 4.2.1 $\pi/3$ -Rotations

As mentioned before,  $\pi/3$  rotations take states representing a node in a projection into each other. It is time to see in detail how these rotations affect a subgraph consisting of a node and its four edges. Fig. 8 shows such a rotation in the case where the node is in a

$\oplus$ -state with respect to the chosen rotation axis before imposing the rotation, while Fig. 9 illustrates the opposite case.

A  $\pi/3$  rotation creates a crossing of two edges of the node and causes twists, which are explicitly labeled, on all edges of the node. The twist number on the rotation axis of a node is always opposite to that of the other three edges of the node. Note that a  $\pi/3$ -rotation changes the state of a node, as shown in Figures 8 and 9, i.e. if the node is in state  $\oplus$  before the rotation, it becomes a  $\ominus$ -node after the rotation. This is the key to the first reason that we put the twists in an edge in units of  $\pi/3$ . A  $\pi/3$  rotation relates two projections of an embedded spin network, which belong to the same diffeomorphism class.

#### 4.2.2 $2\pi/3$ -Rotations

Two consecutive  $\pi/3$  rotations certainly give rise to a  $2\pi/3$  rotation. However, it is intuitive to understand  $2\pi/3$  rotations in a more topological way. Obviously, rotating a tetrahedron by an angle of  $2\pi/3$  with respect to the normal of any of the four faces of the tetrahedron does not change the view of it. Therefore, by the local duality between a node and a tetrahedron, as long as an edge of a node is chosen, the other three edges of the node are on an equal footing. If we rotate a node with respect to any of its four edges by  $2\pi/3$ , the resulting diagram should be diffeomorphic to, or in our context equivalent to, the original one. In Fig. 10 and Fig. 11 we list all the  $2\pi/3$ -rotations.

Each of such rotations generate two crossings and twists on all four edges. The twist number on the rotation axis of a node is always opposite to that of the other three edges of the node. Note that a  $2\pi/3$  rotation does not change the state of a node with respect to the rotation axis, i.e. if a node is in state  $\oplus$  with respect to  $z$  before the rotation, it is still a  $\oplus$ -node after the rotation.

#### 4.2.3 $\pi$ -Rotations

The  $\pi/3$  and  $2\pi/3$  rotations can be used to construct larger rotations, for example the  $\pi$  rotations, which also certainly do not change the diffeomorphism class a projection belongs to. For the convenience of future use, we depict these four possible rotations in Fig. 12 and Fig. 13.

Note that a  $\pi$  rotation changes the state of a node, i.e. if a node is in state  $\oplus$  with respect to its rotation axis before the rotation, it becomes a  $\ominus$ -node with respect to the same axis after the rotation.

$\pi/3$  rotations are the smallest building blocks of all possible rotations; they are thus the generators of all rotations. This is illustrated in Fig. 8 through Fig. 13, each of which can be directly used in a graphic calculation.

Recall that all the equivalence moves defined above are diffeomorphic operations on the embedded graphs. As an example, Fig. 14 depicts two braids that can be deformed

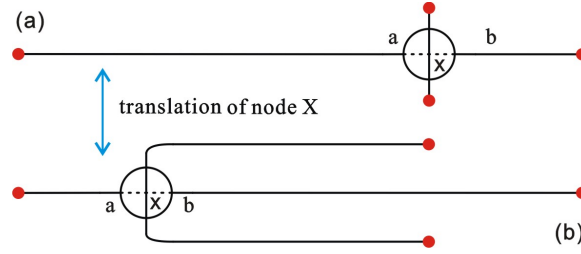


Figure 6: Red points I and J represent other nodes where edges  $a$  and  $b$  are attached to. (b) is obtained from (a) by translating node  $X$  from right to left, and vice versa.

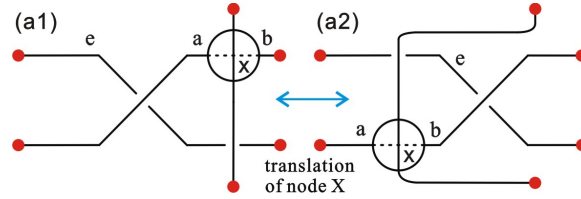


Figure 7: Red points represent other nodes to which edges  $a$  and  $b$  are attached. (a1) and (a2) can be transformed into each other by translating node  $X$ . To transform (b1) into (b2) is not allowed due to the tangle produced by translating node  $X$ .

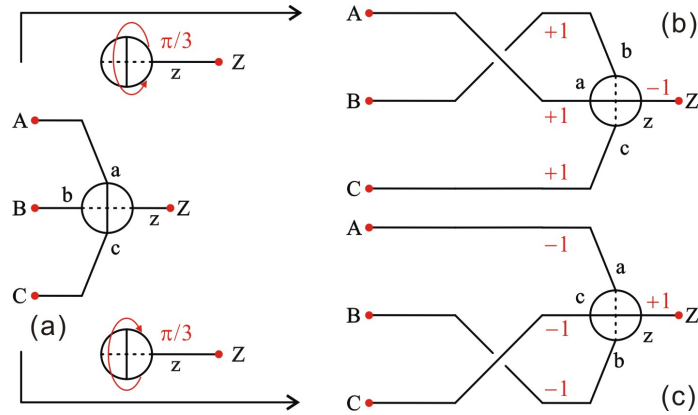


Figure 8: (b) & (c) are results of (a) by rotating the  $\oplus$ -node in (a) w.r.t. edge  $z$  in two directions respectively. Points  $A$ ,  $B$ ,  $C$ , and  $Z$  are assumed to be connected somewhere else and are kept fixed during the rotation. All edges of the node gain the same amount of twist after rotation.

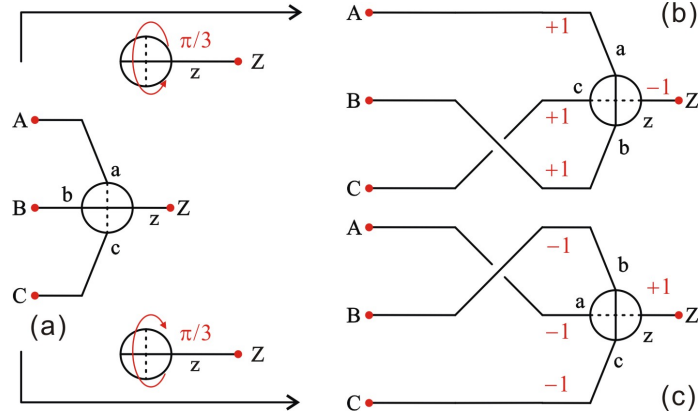


Figure 9: (b) & (c) are results of (a) by rotating the  $\ominus$ -node in (a) w.r.t. edge  $z$  in two directions respectively. Points  $A$ ,  $B$ ,  $C$ , and  $Z$  are assumed to be connected somewhere else and are kept fixed during the rotation. All edges of the node gain the same amount of twist after rotation.

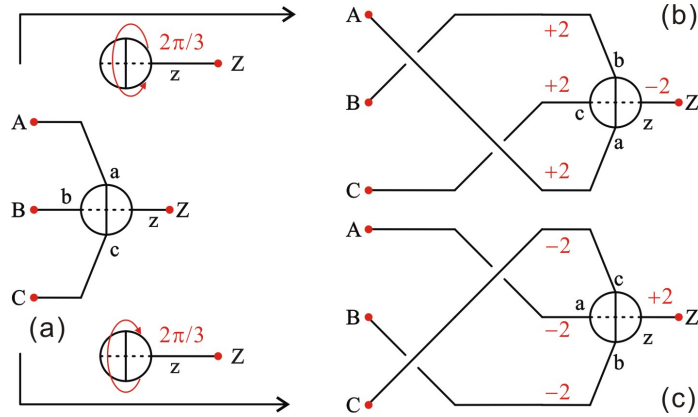


Figure 10: (b) & (c) are results of (a) by rotating the  $\oplus$ -node in (a) w.r.t. edge  $z$  in two directions respectively. Points  $A$ ,  $B$ ,  $C$ , and  $Z$  are assumed to be connected somewhere else and are kept fixed during the rotation. All edges of the node gain the same amount of twist after rotation.

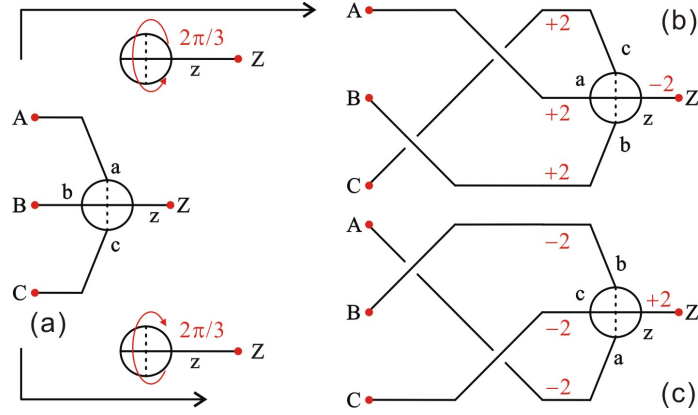


Figure 11: (b) & (c) are results of (a) by rotating the  $\ominus$ -node in (a) w.r.t. edge  $z$  in two directions respectively. Points  $A$ ,  $B$ ,  $C$ , and  $Z$  are assumed to be connected somewhere else and are kept fixed during the rotation. All edges of the node gain the same amount of twist after rotation.

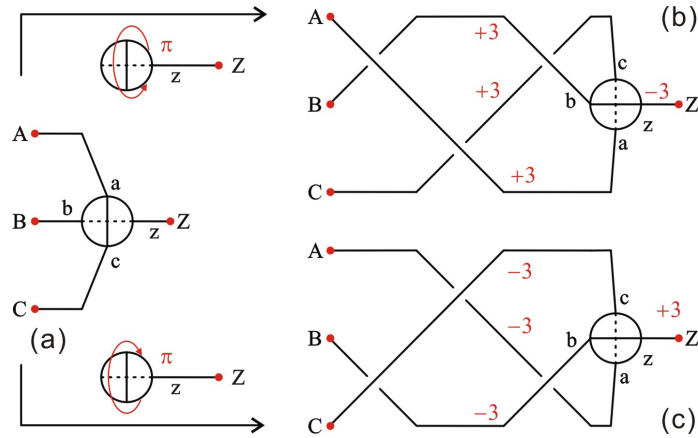


Figure 12: (b) & (c) are results of (a) by rotating the  $\oplus$ -node in (a) w.r.t. edge  $z$  in two directions respectively. Points  $A$ ,  $B$ ,  $C$ , and  $Z$  are assumed to be connected somewhere else and are kept fixed during the rotation. All edges of the node gain the same amount of twist after rotation.

into each other by a  $\pi/3$  rotation of node 2 with respect to its external edge  $z$ . We say these two braids are equivalent to each other.

Note that for an end-node of a braid, only its external edge is allowed to be the rotation axis, with respect to which the equivalence rotation moves are applied. Otherwise, one may end up with a situation similar to Fig. 15, which does not satisfy Definition 1. Therefore, although sub-spinnets like Fig. 15 are equivalent to well-defined braids by rotation moves, they are not to be investigated because they complicate the clear structure of braids and do not have any new interesting property. Thus for simplicity we only allow the external edge of an end-node of a braid to be the rotation axis. If a node is not an end-node of a braid, any of its four edges can be chosen as a rotation axis.

By looking carefully at the rotations and the crossings and twists generated accordingly one can find that the assignment of values to crossings, shown in Fig. 5, is consistent with the assignment of values to twists, shown in Fig. 2.

### 4.3 Conserved quantity: effective twist number

Given that the rotations and translations are well-defined equivalence moves, there should be a conserved quantity, which is the same before and after the moves. Rotations create or annihilate twist and crossings simultaneously, we thus define a composite quantity, christened **effective twist number of a rotation**,

$$\Theta_r = \sum_{e=1}^4 T_e - 2 \times \sum_{\substack{\text{all Xings} \\ \text{created}}} X_i, \quad (1)$$

where  $T_e$  is the twist number created by the rotation on an edge of the node,  $X_i$  is the crossing number of a crossing created by the rotation between any two edges of the node, and the factor of 2 comes from the fact that a crossing always involve two edges. One can easily check that the rotations in Fig. 8 through Fig. 13 satisfy  $\Theta_r \equiv 0$ . That is, rotations have a zero effective twist number. Therefore we can enlarge  $\Theta_r$  to a more general quantity  $\Theta_0$ , the **effective twist number of subdiagrams** of an embedded spinnet, which are related by rotations of nodes, by taking into account all the edges that are affected by rotations. We define

$$\Theta_0 = \sum_{\substack{\text{all edges in a} \\ \text{subdiagram}}} T_e - 2 \times \sum_{\substack{\text{all Xings in a} \\ \text{subdiagram}}} X_i, \quad (2)$$

where  $T_e$  is the twist number on an edge of the subdiagram,  $X_i$  is the crossing number of a crossing in the subdiagram. Since  $\Theta_r \equiv 0$ ,  $\Theta_0$  is indeed a conserved quantity under rotations. Important examples of subdiagrams are braids, which will become clear when we talk about propagation and interactions.

The effective twist number  $\Theta_0$  in Eq. 2 is also found to be preserved by translation moves.

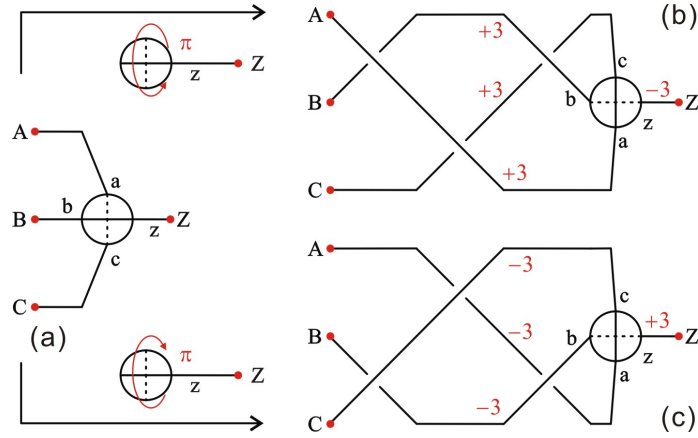


Figure 13: (b) & (c) are results of (a) by rotating the  $\ominus$ -node in (a) w.r.t. edge  $z$  in two directions respectively. Points  $A$ ,  $B$ ,  $C$ , and  $Z$  are assumed to be connected somewhere else and are kept fixed during the rotation. All edges of the node gain the same amount of twist after rotation.

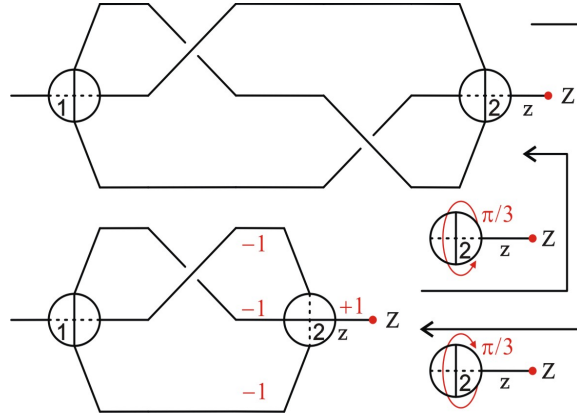


Figure 14: The two braids are equivalent because they can be transformed into each other by a  $\pi/3$ -rotation of node 2.

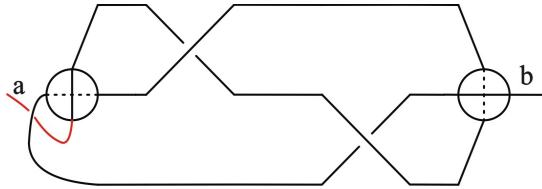


Figure 15: A sub-spinnet equivalent to a braid by rotations with respect to the strands of the braid; it does not satisfy our definition of a 3-strand braid any more.

Note that the effective twist number is not defined in the unframed case, simply because the unframed case has no notion of twists.

## 5 Classification of Braids

With the help of equivalence moves, in particular the rotation moves, we can classify all possible 3-strand braids into two major types, namely reducible braids and irreducible braids, whose definitions are given below. The aforementioned restriction that only the external edge of an end-node of a braid can be the rotation axis of the node ensures the unambiguous assignments of states to the end nodes of a braid and keeps the classification of braids simple. Note that twists on edges are irrelevant to the calculation in the section; they are thus neglected throughout the discussion. Nevertheless, the results are valid for both framed and unframed cases.

For the purpose of classifying the braids, we also consider braids as if they are isolated regions in a graph.

**Definition 2.** A braid is called a **reducible braid**, if it is equivalent to a braid with fewer crossings; otherwise, it is **irreducible**.

The braid on the top part of Fig. 14 is an example of a reducible braid, whereas the one at the bottom of the figure is an irreducible braid. To classify the braids in a convenient way, we need a new notation and some auxiliary definitions. Since we have a way of assigning crossings integers  $+1$  or  $-1$ , as in Fig. 5, we can use  $2 \times N$  matrices with two end-nodes in either state  $\oplus$  or  $\ominus$  to denote a 3-strand braid with  $N$  crossings, as shown in Fig. 16 and its caption, keeping in mind that the state of an end-node is and can only be with respect to its external edge. For the purpose of calculation, it is also convenient to associate crossings with one of the two end-nodes of a braid. For example, in Fig. 16 the left end-node with its nearest crossing can be denoted by  $\oplus \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ , and the right end-node with its nearest two crossings can be written as  $\begin{pmatrix} 0 & 0 \\ +1 & +1 \end{pmatrix} \ominus$ , which has  $\begin{pmatrix} 0 \\ +1 \end{pmatrix} \ominus$  as its 1-crossing **sub end-node**. End-nodes represented in this way are called 1-crossing end-nodes, 2-crossing end-nodes, etc. An end-node without any crossing is christened a **bare end-node**.

A braid can be decomposed into or recombined from a left end-node, a right end-node, and a bunch of crossings represented by matrices. For instance,

$$\oplus \begin{pmatrix} -1 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix} \oplus \iff \oplus \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} +1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix} \oplus,$$

where the "+" between two matrices on the RHS means direct sum or concatenation of two pieces of braids. One can see from the above equation that the first two crossings



or the second and third crossings on the RHS are cancelled. Given this, we have the following definition.

**Definition 3.** An  $N$ -crossing end-node is said to be a **reducible end-node**, if it is equivalent to a  $M$ -crossing end-node with  $M < N$ , by equivalence moves done on the node; otherwise, it is **irreducible**.

The definition above gives rise to another definition of reducible braid, which is equivalent to Definition 2.

**Definition 4.** A braid is said to be **reducible** if it has a reducible end-node. If a braid has a reducible left or right end-node, or both, it is called **left-**, or **right-**, or **two-way-reducible**.

For consistency we may also symbolize the rotation moves. Because rotations are exerted only on the end-nodes of a braid, we can denote all possible moves by rotation operators  $R_{LL}^\theta$ ,  $R_{LR}^\theta$ ,  $R_{RL}^\theta$ , and  $R_{RR}^\theta$ , where the superscript  $\theta$  is the angle of rotation, the first subscript  $L$  ( $R$ ) reads that the operation is on the left (right) end-node of a braid, and the second subscript  $L$  ( $R$ ) indicates that the direction of rotation is left- (right-) handed. The left- (right-) handedness of the rotation is defined in such a way that if you grab the rotation axis of a node in your left (right) hand, with the thumb pointing to the node, your hand wraps up in the direction of rotation. Results of the rotation operators have been shown graphically in Fig. 8 through Fig. 13. Here we show an example of the algebra in the following equation.

$$\begin{aligned}
& R_{RR}^{\pi/3} \left[ \oplus \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & +1 \end{pmatrix} \ominus \right] \\
&= \oplus \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & +1 \end{pmatrix} R_{RR}^{\pi/3} (\ominus) \\
&= \oplus \begin{pmatrix} -1 & 0 & 0 \\ 0 & +1 & +1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \oplus \\
&= \oplus \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & +1 & +1 & 0 \end{pmatrix} \oplus .
\end{aligned}$$

Because a braid can be reduced only from its end-nodes, we first classify the (ir)reducible end-nodes. We start from 1-crossing end-nodes; all the possible ones are listed in table 1

The following equations then show how all the reducible 1-crossing end-nodes are

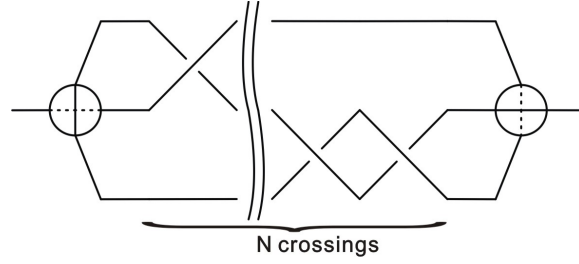


Figure 16: This braid can be represented by  $\oplus \begin{pmatrix} -1 & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & \cdots & +1 & +1 \end{pmatrix} \ominus$ .

Left end-nodes		Right end-nodes
$\oplus \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$		$\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \oplus$
$\oplus \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}$		$\begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \oplus$
$\ominus \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$		$\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \ominus$
$\ominus \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}$		$\begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \ominus$

Table 1: Table of all possible 1-crossing end-nodes.

reduced to bare nodes.

$$\begin{aligned}
R_{LR}^{\pi/3} \left[ \oplus \begin{pmatrix} +1 \\ 0 \end{pmatrix} \right] &= \ominus \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} +1 \\ 0 \end{pmatrix} = \ominus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
R_{LL}^{\pi/3} \left[ \oplus \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] &= \ominus \begin{pmatrix} 0 \\ +1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \ominus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
R_{LL}^{\pi/3} \left[ \ominus \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] &= \oplus \begin{pmatrix} +1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
R_{LR}^{\pi/3} \left[ \ominus \begin{pmatrix} 0 \\ +1 \end{pmatrix} \right] &= \oplus \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ +1 \end{pmatrix} = \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
R_{RL}^{\pi/3} \left[ \begin{pmatrix} -1 \\ 0 \end{pmatrix} \oplus \right] &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} +1 \\ 0 \end{pmatrix} \ominus = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ominus \\
R_{RR}^{\pi/3} \left[ \begin{pmatrix} 0 \\ +1 \end{pmatrix} \oplus \right] &= \begin{pmatrix} 0 \\ +1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \ominus = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ominus \\
R_{RR}^{\pi/3} \left[ \begin{pmatrix} +1 \\ 0 \end{pmatrix} \ominus \right] &= \begin{pmatrix} +1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \oplus = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus \\
R_{RL}^{\pi/3} \left[ \begin{pmatrix} 0 \\ -1 \end{pmatrix} \ominus \right] &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ +1 \end{pmatrix} \oplus = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \oplus
\end{aligned} \tag{3}$$

With the help of the above calculations, we can easily list all the irreducible 1-crossing end-nodes in table 2.

Left end-nodes		Right end-nodes
$\oplus \begin{pmatrix} -1 \\ 0 \end{pmatrix}$		$\begin{pmatrix} +1 \\ 0 \end{pmatrix} \oplus$
$\oplus \begin{pmatrix} 0 \\ +1 \end{pmatrix}$		$\begin{pmatrix} 0 \\ -1 \end{pmatrix} \oplus$
$\ominus \begin{pmatrix} +1 \\ 0 \end{pmatrix}$		$\begin{pmatrix} -1 \\ 0 \end{pmatrix} \ominus$
$\ominus \begin{pmatrix} 0 \\ -1 \end{pmatrix}$		$\begin{pmatrix} 0 \\ +1 \end{pmatrix} \ominus$

Table 2: Table of irreducible 1-crossing end-nodes.

Now we consider 2-crossing end-nodes; there are a total of 48 of this kind, including left and right end-nodes. To find all the irreducible 2-crossing end-nodes, we need only to think about those whose sub 1-crossing nodes are irreducible, since otherwise a 2-crossing end-node is already reducible; this excludes 24 2-crossing end-nodes. If a 2-crossing node has an irreducible sub 1-crossing node, its crossings can definitely not be reduced by  $2\pi/3$ -rotations, because a  $2\pi/3$ -rotation is made of two consecutive  $\pi/3$ -rotations that do

not reduce any irreducible 1-crossing node, and it does not flip the state of a bare node. Interestingly, however, a 2-crossing end-node with an irreducible sub 1-crossing end-node may still be reduced to an irreducible 1-crossing end-node by  $\pi$ -rotations, which can be seen from the following equations.

$$\begin{aligned}
R_{LL}^{\pi} \left[ \oplus \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right] &= \ominus \begin{pmatrix} +1 & 0 & +1 \\ 0 & +1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \ominus \begin{pmatrix} +1 \\ 0 \end{pmatrix} \\
R_{LR}^{\pi} \left[ \oplus \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix} \right] &= \ominus \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix} = \ominus \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\
R_{LR}^{\pi} \left[ \ominus \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} \right] &= \oplus \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} + \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} = \oplus \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\
R_{LL}^{\pi} \left[ \ominus \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \right] &= \oplus \begin{pmatrix} 0 & +1 & 0 \\ +1 & 0 & +1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \oplus \begin{pmatrix} 0 \\ +1 \end{pmatrix} \\
R_{RR}^{\pi} \left[ \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix} \oplus \right] &= \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \ominus = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \ominus \\
R_{RL}^{\pi} \left[ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \right] &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & +1 & 0 \\ +1 & 0 & +1 \end{pmatrix} \ominus = \begin{pmatrix} 0 \\ +1 \end{pmatrix} \ominus \\
R_{LL}^{\pi} \left[ \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \ominus \right] &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} +1 & 0 & +1 \\ 0 & +1 & 0 \end{pmatrix} \oplus = \begin{pmatrix} +1 \\ 0 \end{pmatrix} \oplus \\
R_{LR}^{\pi} \left[ \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} \ominus \right] &= \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \oplus = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \oplus
\end{aligned} \tag{4}$$

Consequently, we can list all the irreducible 2-crossing end-nodes in table 3

Left end-nodes		Right end-nodes	
$\oplus \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$	$\oplus \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \oplus$	$\begin{pmatrix} +1 & +1 \\ 0 & 0 \end{pmatrix} \oplus$
$\oplus \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}$	$\oplus \begin{pmatrix} 0 & 0 \\ +1 & +1 \end{pmatrix}$	$\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \oplus$	$\begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} \oplus$
$\ominus \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$	$\ominus \begin{pmatrix} +1 & +1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \ominus$	$\begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \ominus$
$\ominus \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}$	$\ominus \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \ominus$	$\begin{pmatrix} 0 & 0 \\ +1 & +1 \end{pmatrix} \ominus$

Table 3: Table of irreducible 2-crossing end-nodes.

The following theorem states that there is no need to investigate end-nodes with more crossings to see if they are irreducible.

**Theorem 1.** *An  $N$ -crossing end-node,  $N > 2$ , which has an irreducible 2-crossing sub end-node, is irreducible.*

*Proof.* If the  $N$ -crossing end-node has a irreducible 2-crossing sub end-node, the two crossings nearest to the node are not reducible by either a single  $\pi/3$ - or a single  $2\pi/3$ -rotation on the node. We may consider  $\pi$ -rotations on its 3-crossing sub end-node. However, if a 3-crossing end-node is reducible by a  $\pi$ -rotation, it must contain a reducible 2-crossing sub end-node according to Fig. 12, Fig. 13 and Eq.4, which is contradictory to the condition given in the theorem. This is then true for all cases where  $N > 3$  by simple induction. Therefore, the theorem holds.  $\square$

Equipped with the knowledge of (ir)reducible end-nodes, we are ready to classify braids. The two end-nodes of a braid are either in the same states or in opposite states, we first take a look at braids whose end nodes are in the same states.

**Theorem 2.** All  $N$ -crossing braids in the form  $\oplus \begin{pmatrix} \cdots \\ \cdots \end{pmatrix} \oplus$  and  $\ominus \begin{pmatrix} \cdots \\ \cdots \end{pmatrix} \ominus$  are reducible for  $N \leq 3$ .

*Proof.* It suffices to prove the  $\oplus\oplus$  case, the case of  $\ominus\ominus$  follows similarly or by symmetry.

1)  $N = 1$ . There are only four possibilities, namely  $\oplus \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} \oplus$  and  $\oplus \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \oplus$ ; however, they are all reducible because they all contain one reducible 1-crossing end-node according to Eq. 3.

2)  $N = 2$ . We first consider the braids formed by an irreducible 2-crossing end-node  $\oplus_{irred} (2\text{-crossing})$  or  $(2\text{-crossing}) \oplus_{irred}$ , and a bare end-node  $\oplus_b$ . We do the following decomposition

$$\begin{aligned} \oplus_{irred} (2\text{-crossing}) + \oplus_b &= \oplus_{irred} (2\text{-crossing}) \oplus_b = \oplus_{irred} + (2\text{-crossing}) \oplus_b \\ \oplus_b + (2\text{-crossing}) \oplus_{irred} &= \oplus_b (2\text{-crossing}) \oplus_{irred} = \oplus_b (2\text{-crossing}) + \oplus_{irred}. \end{aligned}$$

Then from table 3, it is readily seen that  $(2\text{-crossing}) \oplus_b$  and  $\oplus_b (2\text{-crossing})$  are always reducible end-nodes for any choice of  $\oplus_{irred} (2\text{-crossing})$  and  $(2\text{-crossing}) \oplus_{irred}$  respectively. That is, the braids formed this way are reducible. We then consider braids formed by two irreducible 1-crossing end-nodes. The first two rows in table 2 and Eq. 4 clearly shows that the result is either an unbraided or one with a reducible 2-crossing end-node.

3)  $N = 3$ . We need only consider braids, each of which is formed by the direct sum of a 2-crossing irreducible end-node and a 1-crossing irreducible end-node. This can be done by taking the direct sum between the (right) left end-nodes in the first two rows of table 2, and the (left) right end-nodes in the first two rows of table 3. It is not hard to see that any resultant braid has merely two possibilities: i) two neighboring crossings are cancelled by the direct sum, which leads to 1-crossing braids that are proven to be reducible in the case of  $N = 1$ ; and ii) a crossing in the irreducible 2-crossing end-node is combined with the irreducible 1-crossing end-node to form a reducible 2-crossing end-node, i.e. the braid is reducible.  $\square$

Theorem 2 does not cover the case where  $N \geq 4$ , which will be included in another theorem soon. Before that, let us consider the braids whose end-nodes are in opposite states.

**Case 1.**  $N$ -crossing braids in the form  $\oplus \begin{pmatrix} \cdots \\ \cdots \end{pmatrix} \ominus$  and  $\ominus \begin{pmatrix} \cdots \\ \cdots \end{pmatrix} \oplus$ , for  $N \leq 3$ . Note that due to Theorem 2, the set of irreducible 1-crossing braids to be found here represents the full set of irreducible braids for  $N \leq 3$ , regardless of the states of the end-nodes.

1.  $N = 1$ . An irreducible braid can only be made by an irreducible 1-crossing end-node and a bare node. From table 2, there are only four options, which are indeed all irreducible; they are now listed in table 4.

$\oplus \begin{pmatrix} -1 \\ 0 \end{pmatrix} \ominus$	$\ominus \begin{pmatrix} +1 \\ 0 \end{pmatrix} \oplus$
$\oplus \begin{pmatrix} 0 \\ +1 \end{pmatrix} \ominus$	$\ominus \begin{pmatrix} 0 \\ -1 \end{pmatrix} \oplus$

Table 4: Table of irreducible 1-crossing braids.

2.  $N = 2$ . It is sufficient to consider the braids formed by an irreducible 2-crossing end-node and a bare end-node in the opposite state. The reason is that if a 2-crossing braid is irreducible, its two 2-crossing end-nodes must be irreducible as well; moreover, if a 2-crossing end-node is irreducible, its 1-crossing sub end-node is already irreducible. Therefore, one can simply add to each irreducible end-node in table 3 a bare end-node in the opposite state to create an irreducible 2-crossing braid. Being a bit redundant, we list all the 16 irreducible 2-crossing braids in table 5.

$\oplus \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \ominus$	$\oplus \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \ominus$	$\ominus \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \oplus$	$\ominus \begin{pmatrix} +1 & +1 \\ 0 & 0 \end{pmatrix} \oplus$
$\oplus \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \ominus$	$\oplus \begin{pmatrix} 0 & 0 \\ +1 & +1 \end{pmatrix} \ominus$	$\ominus \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \oplus$	$\ominus \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} \oplus$
$\ominus \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \oplus$	$\ominus \begin{pmatrix} +1 & +1 \\ 0 & 0 \end{pmatrix} \oplus$	$\oplus \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \ominus$	$\oplus \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \ominus$
$\ominus \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \oplus$	$\ominus \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} \oplus$	$\oplus \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} \ominus$	$\oplus \begin{pmatrix} 0 & 0 \\ +1 & +1 \end{pmatrix} \ominus$

Table 5: Table of irreducible 2-crossing braids.

3.  $N = 3$ . A 3-crossing braid in this case is irreducible if and only if it admits the following two decompositions.

An irreducible 1-crossing end-node + An irreducible 2-crossing end-node  
 An irreducible 2-crossing end-node + An irreducible 1-crossing end-node,

where "+" is understood as the direct sum. The proof of this claim follows immediately from Theorem 1.

It is time to summarize the case of  $N \geq 4$  for  $N$ -crossing braids, regardless of the states of the end-nodes, by the following theorem.

**Theorem 3.** *A  $N$ -crossing braid for  $N \geq 4$  is irreducible, if and only if it admits the decomposition*

*An irreducible 2-crossing end-node  
 + Arbitrary sequence of crossings  
 + An irreducible 2-crossing end-node,*

*where "+" is understood as the direct sum. The only constraint of the arbitrary sequence of crossings is that its last crossing on each side does not cancel the neighboring crossing associated with the end-node on the same side.*

*Proof.* An irreducible 2-crossing end-node contains an irreducible 1-crossing end-node. By theorem 1, if the above decomposition is admitted, the braid is not reducible on either end-node whatever the arbitrary sequence of crossings is up to the constraint. Therefore, the theorem holds.  $\square$

The braids that are interesting to us are those reducible ones, which is shown in the companion paper. Thus we may make more detailed divisions in the type of reducible braids by the definition below.

**Definition 5.** *Given a reducible braid  $B$ , a braid  $B'$  obtained from  $B$  by doing as much reduction as possible is called an **extremum** of  $B$ ;  $B$  may have more than one **extrema**, but all the extrema have the same number of crossings. We then have the following.*

1. *If all extrema of  $B$  are unbraids, i.e. braids with no crossing,  $B$  is said to be **completely reducible**.*
2. *if an extremum of  $B$  can be reached by equivalence moves exerted only on its (left)right end-node,  $B$  is called **extremely (left-)right-reducible**; if  $B$  is also completely reducible,  $B$  is then said to be **completely (left-)right-reducible**. Note that completely (left-) right-reducible implies extremely (left-) right-reducible, but not vice versa in general.*

## 6 Conclusions & Perspectives

In this paper, we proposed a new notation, namely the tube-sphere notation, for embedded (framed) 4-valent spin-networks. By means of this notation, we discovered a type of topological structures, the 3-strand braids, as sub-diagrams of an embedded spin-net. Equivalence moves, including translations and rotations, which divide projections of embeddings of spin-networks into different equivalence classes, are defined and discussed in detail. The equivalence moves are important and useful in two aspects. Firstly, by rotations, we classify 3-strand braids into two major types: reducible braids and irreducible braids, the former of which are further classified for the purpose of subsequent works. Secondly, by equivalence moves one is able to carry out the calculation of braid propagation and interactions of embedded 4-valent spin-nets.

These results serve as foundations for the work in the companion paper and all our future work dealing with braid-like excitations of embedded 4-valent spin-networks. In another paper, we will propose the evolution moves of embedded 4-valent spin-networks, by which some of the (reducible) 3-strand braids are able to propagate on the spin-nets and interact with each other and provide a possible formulation of the dynamics of these local excitations.

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